JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 120 (1997) 187–194

Conjugacy separability of generalized free products of surface groups

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Communicated by C.A. Weibel; received 1 August 1994; revised 28 May 1995

Abstract

We prove that generalized free products of two free groups or finitely generated torsion-free nilpotent groups amalgamating a cyclic subgroup are cyclic conjugacy separable. Using this we prove that generalized free products of two surface groups amalgamating a cyclic subgroup are conjugacy separable. We conjecture that generalized free products of two Fuchsian groups amalgamating a cyclic subgroup are conjugacy separable. © 1997 Elsevier Science B.V.

1991 Math. Subj. Class.: Primary 20E06, 20E26, 20F34; secondary 20F05, 20F10, 57M05

1. Introduction

The fundamental group of a surface of genus n, briefly, the surface group of genus n has a presentation

$$G = \left\langle a_1, b_1, \ldots, a_n, b_n; \prod_{i=1}^{n} [a_i, b_i] \right\rangle .$$

It is easy to see that G is a generalized free product of two free groups amalgamating a cyclic subgroup. Thus applying Dyer's result [3] that generalized free products of two free groups amalgamating a cyclic subgroup are conjugacy separable, it follows immediately that surface groups are conjugacy separable. In [7], Niblo showed that generalized free products of two finitely generated Fuchsian groups are subgroup separable. Thus, in particular, generalized free products of two surface groups amalgamating a cyclic subgroup are subgroup separable. Fine and Rosenberger [4], showed that Fuchsian groups are conjugacy separable. Therefore it is interesting to ask whether generalized free products of two Fuchsian groups amalgamating a cyclic subgroup are

PII S0022-4049(96)00060-6

conjugacy separable. In this paper we show that generalized free products of two surface groups amalgamating a cyclic subgroup are conjugacy separable and conjecture that generalized free products of two Fuchsian groups amalgamating a cyclic subgroup are conjugacy separable.

In Section 2, we introduce the concept of cyclic conjugacy separability and show that generalized free products of two free groups or finitely generated nilpotent groups are cyclic conjugacy separable. Applying this result we prove in Section 3, the conjugacy separability of two surface groups amalgamating a cyclic subgroup.

Throughout this paper terms and notations are standard. For convenience we summarize some of the terms and notations:

$a \sim_G b$ means a, b are conjugates in G.

$\{a\}^G$ denotes the set of all conjugates of a in G.

If $x \in G = A *_H B$ then $\|x\|$ denotes the free product length of x in G.

$N \lhd_f G$ means N is a normal subgroup of finite index in G.

A group G is said to be π_c if for every cyclic subgroup H of G and every $x \in G \backslash H$, there exists $N \lhd_f G$ such that in $\bar{G} = G/N, \bar{x} \notin \bar{H}$.

Let $x, y \in G$ such that $x \not\sim_G y$. Then x, y are said to be *conjugacy distinguishable* if there exists $N \lhd_f G$ such that in $\bar{G} = G/N, \bar{x} \not\sim_{\bar{G}} \bar{y}$. G is said to be *conjugacy separable* if G is conjugacy distinguishable for all $x \not\sim_G y$.

Let $H = \langle h \rangle$ be an infinite cyclic subgroup of G. If for every positive integer n, there exists $N_n \lhd_f G$ such that $N_n \cap H = \langle h^n \rangle$ then G is said to be *H-potent*, briefly, H-pot. A torsion-free group G is said to be *potent* if it is $\langle h \rangle$-pot for all $1 \neq h \in G$.

We also need the following result from [9]:

Theorem 1.1 (Tang [9, Lemma 3.1]). *Let G be a finite extension of a free group or a finitely generated nilpotent group such that G has unique root property for elements of infinite order. Let $x, h \in G$ such that $\{x\}^G \cap \langle h \rangle = \emptyset$. Then there exists $N \lhd_f G$ such that, in $\bar{G} = G/N, \{\bar{x}\}^G \cap \langle \bar{h} \rangle = \emptyset$.*

2. Cyclic conjugacy separability

In this section we introduce the concept of cyclic conjugacy separability. This property turns out to be very useful to study the conjugacy separability of generalized free products of conjugacy separable groups amalgamating a cyclic subgroup. We begin with the following lemma.

Lemma 2.1. *Let G be a finitely generated torsion-free nilpotent group. Let $1 \neq x \in G$. Then for each positive integer n, there exists $N_n \lhd_f G$ such that, in $\bar{G}_n = G/N_n, |\bar{x}| = n$ and $\bar{x}^i \not\sim_{\bar{G}_n} \bar{x}^j$ for $\bar{x}^i \neq \bar{x}^j$.*

Proof. G is a finitely generated torsion-free nilpotent group. This implies there exists positive integer k such that $x \in Z_k(G)$ and $x \notin Z_{k-1}(G)$, where $Z_k(G)$ is the kth upper center of G. Let $\hat{G} = G/Z_{k-1}(G)$. Then $1 \neq \hat{x} \in Z(\hat{G})$. Since $\hat{G}$ is again a finitely generated torsion-free nilpotent group $\hat{x}^i \not\sim_{\hat{G}} \hat{x}^j$ for $i \neq j$. Also $Z(\hat{G})$ is a free abelian group. Thus $Z(\hat{G})$ has a free basis $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_m$ so that $\hat{x} = \hat{y}_1^r$ (say). Let $\hat{N}_n = \hat{G}^{rn}$ and $\bar{G}_n = \hat{G}/\hat{N}_n$. Then $\bar{G}_n$ is finite with $|\bar{x}| = n$. Since $\bar{x} \in Z(\bar{G}_n), \bar{x}^i \not\sim_{\bar{G}_n} \bar{x}^j$ for all $\bar{x}^i \neq \bar{x}^j$. Let N_n be the pre image of $\hat{N}_n$ in the natural homomorphism of G to $\hat{G}$. It is clear N_n is the required normal subgroup of G. □

Corollary 2.2. *Let G be residually free. Let $1 \neq x \in G$. Then for each positive integer n, there exists $N_n \lhd_f G$ such that, in $\bar{G}_n = G/N_n, |\bar{x}| = n$ and $\bar{x}^i \not\sim_{\bar{G}_n} \bar{x}^j$ for $\bar{x}^i \neq \bar{x}^j$.*

Proof. G is residually free implies that there exists $M \lhd G$ such that $x \notin M$ and $\tilde{G} = G/M$ is free. By Magnus' theorem [5], free groups are residually finitely generated torsion-free nilpotent. Therefore there exists $\tilde{L} \lhd \tilde{G}$ such that $\tilde{x} \notin \tilde{L}$ and $\hat{G} = \tilde{G}/\tilde{L}$ is a finitely generated torsion-free nilpotent group. Thus by Lemma 2.1, the result follows. □

We now define the concept of cyclic conjugacy separable.

Definition 2.3. A group G is said to be *cyclic conjugacy separable* if for each $x \in G$ and each cyclic subgroup $\langle h \rangle$ of G such that $\{x\}^G \cap \langle h \rangle = \emptyset$, there exists a finite homomorphic image $\bar{G}$ of G such that $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{h} \rangle = \emptyset$.

Lemma 2.4. *Let $G = A *_H B$, where $H = \langle h \rangle$ with $h \in A,B$ and A,B are π_c, potent and cyclic conjugacy separable. Let $x \in G$ be of minimal length in $\{x\}^G$. Then there exist $M \lhd_f A$ and $L \lhd_f B$ such that $M \cap H = L \cap H$ and $\bar{x}$ is of minimal length in $\{\bar{x}\}^{\bar{G}}$, where $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ with $\bar{A} = A/M$ and $\bar{B} = B/L$.*

Proof. Clearly, if $\|x\| = 0$ the lemma is obvious. Hence we assume $\|x\| \geq 1$.

Case 1: $\|x\| = 1$. Let $x \in A/H$, say. Since x is of minimal length in $\{x\}^G$, this implies $x \not\sim_G h^i$. In particular $x \not\sim_A h^i$. Since A is cyclic conjugacy separable, there exists $M \lhd_f A$ such that, in $\bar{A} = A/M, \{\bar{x}\}^{\bar{A}} \cap \bar{H} = \emptyset$. Now B is potent. Therefore there exists $L \lhd_f B$ such that $M \cap H = L \cap H$. Let $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ where $\bar{B} = B/L$. Then $\bar{x} \not\sim_{\bar{G}} \bar{h}^i$. Hence $\bar{x}$ is of minimal length in $\{\bar{x}\}^{\bar{G}}$.

Case 2: $\|x\| \geq 2$. Since A,B are π_c and potent, there exist $M \lhd_f A$ and $L \lhd_f B$ such that $M \cap H = L \cap H$ and in $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$, where $\bar{A} = A/M$ and $\bar{B} = B/L, \|\bar{x}\| = \|x\|$. Suppose $\bar{y}$ is an element of minimal length in $\{\bar{x}\}^{\bar{G}}$. Then $\bar{x} \sim_{\bar{G}} \bar{y}$. Since x is of minimal length in $\{x\}^G$, x is cyclically reduced. Thus, by the choice of M and $L, \bar{x}$ is cyclically reduced in $\bar{G}$. Hence, by Solitar's theorem [6, p. 212], $\bar{x} \sim_{\bar{G}} \bar{y}$ implies $\|\bar{x}\| = \|\bar{y}\|$. It follows that $\bar{x}$ is also of minimal length in $\{\bar{x}\}^{\bar{G}}$. □

In [3, Lemmas 6 and 8] Dyer showed that finitely generally nilpotent groups and free groups are cyclic conjugacy separable. Applying Theorem 1.1, we make the following useful observation.

Theorem 2.5. *Let $G = A *_H B$ where A, B are finite groups. Then G is cyclic conjugacy separable.*

Proof. We note that G is a free-by-finite group and G has unique root property for elements of infinite order. Hence, by Theorem 1.1, G is cyclic conjugacy separable. $\square$

In order to prove generalized free products of free groups and finitely generated nilpotent groups amalgamating a cyclic subgroup are cyclic conjugacy separable we need the following result.

Lemma 2.6. *Let $G = A *_H B$ be conjugacy separable, where $H = \langle h \rangle$, and A, B are H-separable and H-potent. Let $c \in G$ such that c is cyclically reduced and $\|c\| \geq 2$. If $x \in G$ such that $\{x\}^G \cap \langle c \rangle = \emptyset$, then there exists $N \lhd_f G$ such that, in $\bar{G} = G/N, \{\bar{x}\}^{\bar{G}} \cap \langle \bar{c} \rangle = \emptyset$.*

Proof. We can assume x to be of minimal length in $\{x\}^G$. Also let $c = a_1 b_1 \ldots a_n b_n$, say, where $a_i \in A \backslash H$ and $b_i \in B \setminus H$.

Case 1: $\|x\| \leq 1$. Let $x \in A$, say. Since A, B are residually finite, H-separable and H-potent, there exist $M \lhd_f A$ and $L \lhd_f B$ such that $x \notin M$, $a_i \notin MH$, $b_i \notin LH$ and $M \cap H = M \cap L$. Let $\tilde{G} = \tilde{A} *_{\tilde{H}} \tilde{B}$, where $\tilde{A} = A/M$ and $\tilde{B} = B/L$. Then $1 \neq \tilde{x} \in \tilde{A}$ and $\tilde{c} = \tilde{a}_1 \tilde{b}_1 \ldots \tilde{a}_n \tilde{b}_n$. This implies that any conjugate of $\tilde{x}$ is either an element of $\tilde{A} \cup \tilde{B}$ or of the form $\tilde{u}^{-1} \tilde{y} \tilde{u}$ where $\tilde{u}^{-1} \tilde{y} \tilde{u}$ and $\tilde{u}$ are reduced words with $\tilde{y} \in \tilde{A} \cup \tilde{B}$. Since c is cyclically reduced, it follows that $\tilde{u}^{-1} \tilde{y} \tilde{u} \neq \tilde{c}^i$ for any integer i. Therefore $\{\tilde{x}\}^{\tilde{G}} \cap \langle \tilde{c} \rangle = \emptyset$. Now $\tilde{A}$ and $\tilde{B}$ are finite. Thus, by Theorem 2.5, $\tilde{G}$ is cyclic conjugacy separable. It follows immediately there exists $N \lhd_f G$ such that, in $\bar{G} = G/N, \{\bar{x}\}^{\bar{G}} \cap \langle \bar{c} \rangle = \emptyset$.

Case 2: $\|x\| \geq 2$. $\{x\}^G \cap \langle c \rangle = \emptyset$ implies $x^* \neq c^i$, where x^* is any cyclic permutation of x. Let k be a positive integer such that $\|c^k\| > \|x\|$. Since x has only a finite number of cyclic permutations and A, B are residually finite, H-separable and H-potent, there exist $M \lhd_f A$ and $L \lhd_f B$ such that, in $\tilde{G} = \tilde{A} *_{\tilde{H}} \tilde{B}$, where $\tilde{A} = A/M$ and $\tilde{B} = B/L$ we have $\|\tilde{x}\| = \|x\|, \|\tilde{c}\| = \|c\|$ and $\tilde{x}^* \neq \tilde{c}^i$ for all $0 < |i| < k$, where k is the smallest integer such that $\|\tilde{c}^k\| > \|\tilde{x}\|$. Since G is conjugacy separable, there exists $K \lhd_f G$ such that, in $\hat{G} = G/K$, $\hat{x} \not\sim_{\hat{G}} \hat{c}$. Let $M_1 = M \cap K$ and $L_1 = L \cap K$. Since A, B are H-potent we can assume $M_1 \cap H = L_1 \cap H$. Let $\bar{\bar{G}} = \bar{\bar{A}} *_{\bar{\bar{H}}} \bar{\bar{B}}$, where $\bar{\bar{A}} = A/M_1$ and $\bar{\bar{B}} = B/L_1$. Then $\| \bar{\bar{x}} \| = \|x\|, \| \bar{\bar{c}} \| = \|c\|, \bar{\bar{x}}^* \neq \bar{\bar{c}}^i$ for $0 < |i| < k$ and $\bar{\bar{x}} \not\sim_{\bar{\bar{G}}} \bar{\bar{c}}$. Moreover, $\bar{\bar{x}}$ and $\bar{\bar{c}}$ are cyclically reduced. Let $\bar{\bar{g}} \in \bar{\bar{G}}$. If $\bar{\bar{g}}^{-1} \bar{\bar{x}} \bar{\bar{g}}$ is a cyclic permutation of $\bar{\bar{x}}$ then, by the choice of M_1 and L_1, $\bar{\bar{g}}^{-1} \bar{\bar{x}} \bar{\bar{g}} \neq \bar{\bar{c}}^i$ for $0 < |i| < k$. Moreover, $\bar{\bar{g}}^{-1} \bar{\bar{x}} \bar{\bar{g}} \neq \bar{\bar{c}}^i$ for $i \geq k$, since $\| \bar{\bar{c}}^i \| > \| \bar{\bar{g}}^{-1} \bar{\bar{x}} \bar{\bar{g}} \|$, whence $\bar{\bar{g}}^{-1} \bar{\bar{x}} \bar{\bar{g}} \neq \bar{\bar{c}}^i$ for all i. If $\bar{\bar{g}}^{-1} \bar{\bar{x}} \bar{\bar{g}}$ is not a cyclic permutation of $\bar{\bar{x}}$ then $\bar{\bar{g}}^{-1} \bar{\bar{x}} \bar{\bar{g}}$ has the reduced form $\bar{\bar{u}}^{-1} \bar{\bar{w}} \bar{\bar{u}}$, where $\bar{\bar{u}} \in \bar{\bar{A}} \setminus \bar{\bar{H}}$ or $\bar{\bar{B}} \setminus \bar{\bar{H}}$. This implies $\bar{\bar{g}}^{-1} \bar{\bar{x}} \bar{\bar{g}} \neq \bar{\bar{c}}^i$ for all i. Thus, in $\bar{\bar{G}}$, $\{\bar{\bar{x}}\}^{\bar{\bar{G}}} \cap \langle \bar{\bar{c}} \rangle = \emptyset$. Since $\bar{\bar{A}}, \bar{\bar{B}}$ are finite, by Theorem 2.5, there exists $N \lhd_f G$ such that, in $\bar{G} = G/N$, $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{c} \rangle = \emptyset$. $\square$

We are now ready to prove the cyclic conjugacy separability of generalized free products of free groups or finitely generated nilpotent groups amalgamating a cyclic subgroup.

Theorem 2.7. *Let $G = A *_H B$, where $H = \langle h \rangle$ and A, B are free or finitely generated nilpotent groups. Then G is cyclic conjugacy separable.*

Proof. We shall only prove the case when A, B are free. The proof for the nilpotent case is similar since the additional consideration for elements of finite orders can be easily looked after.

Let $x, c, \in G$ such that $\{x\}^G \cap \langle c \rangle = \emptyset$. Since $\{x\}^G \cap \langle c \rangle = \emptyset$ if and only if $\{x\}^G \cap \langle g^{-1}cg \rangle = \emptyset$ for any $g \in G$, we can assume c to be cyclically reduced.

Case 1: $\|c\| = 0$.

(a) $x \in A \cup B$. Say, $x \in A$. By Dyer [3], there exists $M \lhd_f A$ such that, in $\tilde{A} = A/M, \{\tilde{x}\}^{\tilde{A}} \cap \langle \tilde{c} \rangle = \emptyset$. Since B is potent, there exists $L \lhd_f B$ such that $L \cap H = M \cap H$. Moreover, by Corollary 2.2, we can choose L such that, in $\tilde{B} = B/L$, $\tilde{h}^i \not\sim_{\tilde{B}} \tilde{h}^j$ for $\tilde{h}^i \neq \tilde{h}^j$. Let $\tilde{G} = \tilde{A} *_{\tilde{H}} \tilde{B}$. Then $\{\tilde{x}\}^{\tilde{G}} \cap \langle \tilde{c} \rangle = \emptyset$. Since $\tilde{A}$ and $\tilde{B}$ are finite, by Theorem 2.5, $\tilde{G}$ is cyclic conjugacy separable. It follows that there exists $N \lhd_f G$ such that, in $\bar{G} = G/N, \{\bar{x}\}^{\bar{G}} \cap \langle \bar{c} \rangle = \emptyset$.

(b) $x \notin A \cup B$. We can assume x to be of minimal length in $\{x\}^G$ and $\|x\| \geq 2$. Since A, B are free, whence A, B are H-separable and H-potent, there exist $M \lhd_f A$ and $L \lhd_f B$ such that $M \cap H = L \cap H$ and, in $\tilde{G} = \tilde{A} *_{\tilde{H}} \tilde{B}$, where $\tilde{A} = A/M$ and $\tilde{B} = B/L$, we have $\|\tilde{x}\| = \|x\|$ and $\tilde{c} \neq 1$. Now, by Lemma 2.4, we can assume $\tilde{x}$ to be of minimal length in$\{\tilde{x}\}^{\tilde{G}}$. It follows that $\{\tilde{x}\}^{\tilde{G}} \cap \langle \tilde{c} \rangle = \emptyset$. Thus, as in (a), applying Theorem 2.5, there exists $N \lhd_f G$ such that, in $\bar{G} = G/N, \{\bar{x}\}^{\bar{G}} \cap \langle \bar{c} \rangle = \emptyset$.

Case 2: $\|c\| = 1$. Let $c \in A \backslash H$, say. If $x \in A$, we can use the same argument as in Case 1. Hence we can assume $x \in B \backslash H$. Since A is subgroup separable, there exists $M \lhd_f A$ such that, in $\tilde{A} = A/M, \tilde{c} \notin \tilde{H}$. Now B is free. By Dyer [3], there exists $L \lhd_f B$ such that in $\tilde{B} = B/L$, $\{\tilde{x}\}^{\tilde{B}} \cap \langle \tilde{c} \rangle = \emptyset$. Moreover, by the potency of A and B, we can assume $M \cap H = L \cap H$. Let $\tilde{G} = \tilde{A} *_{\tilde{H}} \tilde{B}$. Then $\{\tilde{x}\}^{\tilde{G}} \cap \langle \tilde{c} \rangle = \emptyset$. Since $\tilde{A}$ and $\tilde{B}$ are finite, by Theorem 2.5, there exists $N \lhd_f G$ such that, in $\bar{G} = G/N, \{\bar{x}\}^{\bar{G}} \cap \langle \bar{c} \rangle = \emptyset$.

Case 3: $\|c\| \geq 2$. Applying Lemma 2.6, the result follows immediately.

This completes the proof. □

3. Main result

We begin by proving some properties of the surface groups that will be needed to prove that the generalized free products of surface groups amalgamating a cyclic subgroup are conjugacy separable.

Lemma 3.1. *Let G be a surface group. Let $1 \neq x \in G$. Then $x^i \not\sim x^j$ for $i \neq j$.*

Proof. By Baumslag [2], G is residually free. This implies that there exists $M \lhd_f G$ such that $x \notin M$ and $\bar{G} = G/M$ is free. Now, by Magnus' theorem [5], $\bar{G}$ is a residually finitely generated torsion-free nilpotent. Therefore, there exists $\bar{N} \lhd \bar{G}$ such that $\bar{x} \notin \bar{N}$ and $\hat{G} = \bar{G}/\bar{N}$ is a finitely generated torsion-free nilpotent group. Thus, if $x^i \sim_G x^j$ for $i \neq j$ then $\hat{x}^i \sim_{\hat{G}} \hat{x}^j$ for $i \neq j$. But this is impossible since $\hat{G}$ is a finitely generated torsion-free nilpotent group. Hence $x^i \not\sim_G x^j$ for $i \neq j$. □

Lemma 3.2. *Let G be a surface group. Let $u, v, x \in G$ such that $u, v \notin \langle x \rangle$. If there exists a pair (k, l) of integers such that $u = x^k v x^l$ then such a pair is unique.*

Proof. Suppose (i, j) is another pair such that $u = x^i v x^j$. Then $x^i v x^j = x^k v x^l$. Thus $v^{-1} x^{i-k} v = x^{l-j}$. By Lemma 3.1, this is not possible unless $i - k = l - j$. Therefore, $v^{-1} x^{i-k} v = x^{i-k}$. Since surface groups has unique root property, we have $v^{-1} x v = x$. But this is not possible because G is residually free and $v \notin \langle x \rangle$. Hence (k, l) is unique. □

In [10], You proved that surface groups are product separable, that is, if $H_1, H_2, \ldots, H_n$ are finitely generated subgroups of a surface group G then, for each $x \in G \backslash H_1 H_2 \ldots H_n$, there exists $N_x \lhd_f G$ such that in $\bar{G} = G/N$, $\bar{x} \notin \bar{H}_1 \bar{H}_2 \ldots \bar{H}_n$. Since finitely generated Fuchsian groups are finite extensions of surface groups, it follows that finitely generated Fuchsian groups are product separable. Applying You's result it is easy to see that finitely generated Fuchsian groups, whence surface groups, are double coset separable.

Definition 3.3. A group G is said to be *double coset separable* if for any two finitely generated subgroups H, K of G and for any $x, g \in G$ such that $x \notin HgK$, there exists $N_{x,g} \lhd_f G$ such that, in $\bar{G} = G/N_{x,g}$, $\bar{x} \notin \bar{H}\bar{g}\bar{K}$.

Lemma 3.4. *Finitely generated Fuchsian groups are double coset separable.*

Proof. Let G be a finitely generated Fuchsian group. Let H, K be any finitely generated subgroups of G. We note that, for $x, g \in G, x \notin HgK$ if and only if $xg^{-1} \notin H(gKg^{-1})$. Since gKg^{-1} is again a finitely generated subgroup of G, by You [10], G is $H(gKg^{-1})$-separable. Hence G is double coset separable. □

Corollary 3.5. *Surface groups are double coset separable.*

Remark. Niblo [8] has also proved the above result with a longer proof.

We are now ready to prove our main result.

Theorem 3.6. *The generalized free products of two surface groups amalgamating a cyclic group is conjugacy separable.*

Proof. Let A,B be surface groups and $G = A *_H B$, where $H = \langle h \rangle$. Let $x, y \in G$ such that $x \not\sim_G y$. We can assume x, y to be of minimal length in $\{x\}^G$ and $\{y\}^G$, respectively.

Case 1: $\|x\| = \|y\| = 0$. This means $x = h^i$, $y = h^j$ and $i \neq j$. Since surface groups are residually free, by Corollary 2.2, there exist $N \lhd_f A$ and $M \lhd_f B$ such that, in $\bar{A} = A/N, \bar{B} = B/M, \bar{h}^i \not\sim_{\bar{A}} \bar{h}^j$ and $\bar{h}^i \not\sim_{\bar{B}} \bar{h}^j$ for $\bar{h}^i \neq \bar{h}^j$. Since Corollary 2.2 applies to every positive integer n, we can assume $N \cap H = M \cap H$. It follows that $\bar{h}^i \not\sim_{\bar{G}} \bar{h}^j$ for $\bar{h}^i \neq \bar{h}^j$ in $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$. Since A,B are finite, by Dyer [3], $\bar{G}$ is conjugacy separable. Hence x, y are conjugacy distinguishable.

Case 2: $\|x\| \neq \|y\|$ and $\|x\|, \|y\| \leq 1$. Let $\|x\| = 0$ and $\|y\| = 1$, say. Since y is of minimal length in $\{y\}^G, \{y\}^B \cap H = \emptyset$. By Theorem 2.7, there exists $M \lhd_f B$ such that, in $\bar{B} = B/M, \{\bar{y}\}^{\bar{B}} \cap \bar{H} = \emptyset$. Thus $\bar{x} \not\sim_{\bar{B}} \bar{y}$. Now surface groups are generalized free products of two free groups amalgamating a cyclic subgroup. Therefore, by Allenby [1], they are potent. This implies that there exists $N \lhd_f A$ such that $N \cap H = M \cap H$. Let $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$, where $\bar{A} = A/N$ and $\bar{B} = B/M$. Then $\bar{x} \not\sim_{\bar{G}} \bar{y}$. Since $\bar{A}, \bar{B}$ are finite, again by Dyer [3], $\bar{G}$ is conjugacy separable. Hence x, y are conjugacy distinguishable in G.

Case 3: $\|x\|, \|y\| \geq 2$ and $\|x\| \neq \|y\|$. By Niblo [7], A,B are subgroup separable. Thus, in particular, A,B are H-separable. Moreover, A,B are H-pot. Therefore, there exist $N \lhd_f A$, $M \lhd_f B$ such that, in $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$, $\|\bar{x}\| = \|x\|$ and $\|\bar{y}\| = \|y\|$. Thus $\|\bar{x}\| \neq \|\bar{y}\|$. By Theorem 2.7, A,B are cyclic conjugacy separable. It follows from Lemma 2.4 that $\bar{x}$ and $\bar{y}$ are of minimal length in $\{\bar{x}\}^{\bar{G}}$ and $\{\bar{y}\}^{\bar{G}}$, respectively. This implies $\bar{x} \not\sim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable, if follows that x, y are conjugacy distinguishable in G.

Case 4: $\|x\| = \|y\| \geq 2$. Let $x = u_1 u_2 \ldots u_r$ and $y = v_1 v_2 \ldots v_r$ be reduced words in G. By Dyer [3], $x \sim_G y$ if and only if for some positive integer i, $1 \leq i \leq r$, the system of equations:

$$\left.\begin{aligned} u_{i+1} &= x_0^{-1} v_1 x_1 \\ u_{i+2} &= x_1^{-1} v_2 x_2 \\ &\cdots \\ &\cdots \\ u_{i+r} &= x_{r-1}^{-1} v_r x_0 \end{aligned}\right\} I(i)$$

has a solution in H.

If for each $i, 1 \leq i \leq r, I(i)$ having no solution in H implies that there exists an integer $k, 1 \leq k \leq r$, such that $u_{i+k} \notin H v_k H$, then by Corollary 3.5, there exists $N \lhd_f A$ such that in $\bar{A} = A/N, \bar{u}_{i+k} \notin \bar{H} \bar{v}_k \bar{H}$ if $u_{i+k}, v_k \in A$. In the same way, there exists $M \lhd_f B$ such that, in $\bar{B} = B/M$, $\bar{u}_{i+k} \notin \bar{H} \bar{v}_k \bar{H}$ if u_{i+k}, $v_k \in B$. Moreover, since A,B are potent we can choose M,N such that $M \cap H = M \cap H$. Let $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$. Then $\bar{x} \not\sim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable, it follows that x, y are conjugacy distinguishable in G. Hence we can assume $I(i)$ has no solution for each i, but for some i, each

$$u_{i+j} = x_{j-1}^{-1} v_j x_j, \tag{3.1}$$

where $1 \le j \le r$, has a solution in H, say, $u_{i+j} = h^{\alpha_j} v_j h^{\beta_j}$. By Lemma 3.2, the solutions are unique for each equation. Thus, for each i, there are only finitely many solutions for x_j's. Since $x \not\sim_G y$, no combination of the solutions to (3.1) is a solution for $I(i)$ for each $1 \le i \le r$. Since A, B are potent, we can arrange $N \lhd_f A$ and $M \lhd_f B$ such that $N \cap H = M \cap H = \langle h^t \rangle$, where $t > 2m$ with m greater than the maximum of the $|\alpha_k|$'s for which h^{α_k} is a solution to (3.1) for $1 \le i,\ j \le r$. It follows that $\overline{I(i)}$ has no solution in $\bar{H}$ in $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ for each $1 \le i \le r$, where $\bar{A} = A/N$ and $\bar{B} = B/M$. Hence $\bar{x} \not\sim_{\bar{G}} \bar{y}$. Since $\bar{G}$ is conjugacy separable, it follows that x, y are conjugacy distinguishable. This completes the proof. □

Conjecture. Since finitely generated Fuchsian groups are finite extensions of surface groups, we conjecture that the generalized free products of two finitely generated Fuchsian groups amalgamating a cyclic subgroup are conjugacy separable.

Acknowledgements

The author gratefully acknowledges the partial support by the National Science and Engineering Research Council of Canada, Grant No. A-6064

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